MATH 210: Introduction to Analysis

Spring 2015-2016, Midterm 1, Duration: 60 min.

Exercise 1.

(a) (**15 points**) Let $\{x_n\}$ and $\{y_n\}$ be two sequences of real numbers converging respectively to l_1 and l_2. Prove that $\{x_n y_n\}$ converges to $l_1 l_2$.
(b) (**7 points**) Prove that if $\{x_n\}$ is bounded and if $\{y_n\}$ converges to 0 then $\{x_n y_n\}$ converges to 0.
(c) (**3 points**) Find an example to illustrate that if the sequence $\{x_n\}$ in (b) is not bounded then the conclusion of (b) might fail.
(d) (**3 points**) Find an example to illustrate that if the sequence $\{y_n\}$ in (b) converges to $l \neq 0$ then the conclusion of (b) might fail.

Exercise 2.

(a) (**10 points**) Prove <u>using the definition of convergence</u> that the sequence $\left\{\dfrac{n+\frac{1}{n}}{n+1}\right\}$ converges to 1.
(b) Consider the sequence $\{(-1)^n\}$
  i. (**7 points**) Prove <u>using the definition of convergence</u> that $\{(-1)^n\}$ diverges.
  ii. (**5 points**) Use another method (of your choice) to show that $\{(-1)^n\}$ diverges.

Exercise 3. Consider the sequence $\{x_n\}$ with $x_n = (-1)^n + \dfrac{1}{n}$.

(a) (**5 points**) Find the $\overline{\lim}$ of $\{x_n\}$. Detail you answer for full credits
(b) (**5 points**) Find the $\underline{\lim}$ of $\{x_n\}$. Detail you answer for full credits
(c) (**5 points**) Deduce from (a) and (b) that $\{x_n\}$ does not converge.
(d) (**5 points**) Let $\{x_{n_k}\}$ be a convergent subsequence of $\{x_n\}$. Prove that the limit of $\{x_{n_k}\}$ is either equal to $\overline{\lim}$ of $\{x_n\}$ or to $\underline{\lim}$ of $\{x_n\}$.

Exercise 4. A fix point theorem. Let $f : [0,1] \to [0,1]$ be an increasing function, that is if $x \leq y$ then $f(x) \leq f(y)$. Consider the set

$$A = \{x \in [0,1] \mid f(x) \leq x\}.$$

(a) (**5 points**) Show that $1 \in A$.
(b) (**5 points**) Prove that A is bounded below.
(c) (**5 points**) Prove that $\inf A$ exists. Denote $\alpha := \inf A$.
(d) (**5 points**) Prove that if $x \in A$ then $f(x) \in A$.
(e) Assume by contradiction that $\alpha < f(\alpha)$.
  i. (**5 points**) Show that there is an element $a \in A$ such that $\alpha \leq a < f(\alpha)$.
  ii. (**3 points**) Prove that $a < f(a)$.
  iii. (**2 points**) Deduce that $f(\alpha) \leq \alpha$.
(f) (**2 points**) Deduce from (e) that $\alpha \in A$.
(g) (**3 points**) Conclude that $f(\alpha) = \alpha$.